

# MATRIX ELEMENTS OF LIGHT QUARKS

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Lattice results for matrix elements of light quark operators are reviewed. The discussion is focused on recent theoretical progress and new numerical calculations which appeared after the Lattice 2000 Conference.

## 1. Introduction

In this talk a review of theoretical and numerical results for matrix elements of light quark operators is presented. The review is focused on the definition, renormalization and improvement of operators relevant to different processes and on the evaluation of the corresponding physical amplitudes. At this Conference the main topics covered in the parallel sessions were electromagnetic and weak form factors, structure functions,  $K^0-\bar{K}^0$  mixing, kaon decay amplitudes and light-cone wave functions. Correspondingly the problem of the renormalization and improvement of the operators was also discussed. Results with perturbative and non-perturbative, gauge invariant or gauge dependent renormalization methods, using Ward identities, with unimproved or improved operators, with Wilson like (improved, twisted, . . .), Domain Wall or overlap fermions were presented. In total I counted 38 talks in the parallel sessions and 13 posters. It is clear that it would have been impossible to discuss all this material, to which two accurate and complete review talks by L. Lellouch [1] and S. Sint [2] were dedicated at Bangalore. A selection of the subjects has been then unavoidable and I apologize to the authors of submitted contributions and to the speakers of talks given in the parallel sessions which I had to omit in my review. I also warmly thank all colleagues who have kindly sent material and information for the preparation of this talk.

The most exciting news at this Conference are the intriguing results for  $\epsilon'/\epsilon$  presented by the CP-PACS [3] and RBC [4] Collaborations. Thus I start from these results to discuss a few physics issues of general interest. The renormalization of lattice quark operators, the *ultraviolet* problem, and the calculation of final state interaction effects, the *infrared* problem, which is essential for kaon decays, are the next subjects. This will give me the opportunity to show several results that were presented in parallel talks at this Conference. More physics problems which are interesting for the lattice community can be found in the talk by M. Beneke at this Conference [5] and by J. Donoghue at the Kaon 2001 Conference [6].

The main topics in this review are: i) the determination of the  $\Delta I = 1/2$  and  $\Delta I = 3/2$   $K \rightarrow \pi\pi$  amplitudes [3,4,7,8]; ii) the main contributions to  $\epsilon'/\epsilon$  due to the electropenguin operators ( $Q_7$  and  $Q_8$  in the standard notation) and to the strong penguin operator  $Q_6$  [3,4,7,8]; iii) matrix elements of SUSY operators which can be computed on the lattice [9]; iv) studies of the chiral expansion of the relevant amplitudes in finite and infinite volumes, in the quenched and unquenched cases [10]–[13]; v) studies of Final State Interactions (FSI) in lattice simulations [8]; vi) the extraction of the physical amplitudes from matrix elements computed in the Euclidean space-time on a finite volume [14,15]. The problem of the renormalization and improvement of the relevant operators will be mentioned and discussed along with the presentation of the different arguments. Although the discussion is mainly focused on kaon physics, many considerations are general and apply to the calculation of all matrix elements.

## 2. General physics issues

Flavour changing neutral currents (FCNC) have been and are the main source of information on quark masses, couplings and CP violation. It is sufficient to mention that they led to the GIM theory and that the large  $B-\bar{B}$  mixing was the first strong indication that the top quark was very heavy, long before precision tests at LEP or its discovery. They are also a window on physics beyond the Standard Model (SM). Since in the SM FCNC cannot occur at tree level, their effects compete with contributions from other heavy sectors of the theory as the extra particles predicted, for example, by Supersymmetry. This explains the large experimental and theoretical efforts in FCNC studies. In table 1, a list of FCNC processes for which experimental measurements exist is presented. In two cases the comparison of the theory with experiments requires the knowledge of the hadronic matrix elements. Lattice results for the matrix element of the  $\Delta S = 2$  operator relevant for  $\epsilon$ , encoded in the bag parameter  $\hat{B}_K$ , exist since a long time. This year, for the first time, we have two complete calculations of the matrix elements of the operators entering  $\epsilon'/\epsilon$  (and also  $ReA_0$  and  $ReA_2$ ) which

Table 1

Experimental measurements and theoretical predictions for several neutral current processes: the CP violation parameters  $\epsilon$  and  $\epsilon'/\epsilon$ , the neutral  $B$  meson mixing  $\Delta M_s$  and  $\Delta M_d$ , radiative inclusive  $b \rightarrow s\gamma$  decays and  $K^+ \rightarrow \pi^+\nu\bar{\nu}$ . In the latter two cases the calculation can be done by using the HQET expansion and the experimental  $K \rightarrow \pi$  weak current matrix element respectively. Thus a lattice calculation of the hadronic amplitudes is not needed.

Quantity	Experiments	Theory
$\epsilon$	$2.271 \pm 0.017 \times 10^{-3}$ [16]	$\eta(1-\rho)B_K$
$\epsilon'/\epsilon$	$17.2 \pm 1.8 \times 10^{-4}$ [17,18]	$-7 \div 30 \times 10^{-4}$
$\Delta M_s/\Delta M_d$	$> 30$ (95% C.L.)	$[(1-\rho)^2 + \eta^2]^{-1}\xi$
$BR(B \rightarrow X_s\gamma)$	$3.11 \pm 0.39 \times 10^{-4}$ [19]	$3.50 \pm 0.50 \times 10^{-4}$ No Lattice Needed
$BR(K^+ \rightarrow \pi^+\nu\bar{\nu})$	$1.5^{+3.4}_{-1.2} \times 10^{-10}$ [20]	$0.8 \pm 0.3 \times 10^{-10}$ No Lattice Needed

Table 2

Lattice results for  $\Delta I = 1/2$  transitions using  $K \rightarrow \pi$  matrix elements from RBC and CP-PACS. The experimental numbers are also given.

Reference	$Re\mathcal{A}_0$	$Re\mathcal{A}_2$	$Re\mathcal{A}_0/Re\mathcal{A}_2$	$\epsilon'/\epsilon$
CP-PACS [3]	$16 \div 21 \times 10^{-8}$	$1.3 \div 1.5 \times 10^{-8}$	$9 \div 12$	$-7 \div -2 \times 10^{-4}$
RBC [4]	$29 \div 31 \times 10^{-8}$	$1.1 \div 1.2 \times 10^{-8}$	$24 \div 27$	$-8 \div -4 \times 10^{-4}$
Exps. [16–18] 2001	$33.3 \times 10^{-8}$	$1.5 \times 10^{-8}$	22.2	$17.2 \pm 1.8 \times 10^{-4}$

can be compared with the experimental results. Here I want first to examine the physics implications of the RBC and CP-PACS calculations which are in total disagreement with the experimental measurements of  $\epsilon'/\epsilon$  by NA48 and KTeV, table 2. If confirmed, these results imply other contributions to  $\epsilon'/\epsilon$  from physics beyond the SM. Let me give an explicit example which could explain the discrepancy. In the effective Hamiltonian, besides the dimension-six four fermion operators considered in refs. [3,4], there are two dimension-five chromomagnetic operators of the form

$$\mathcal{H}_g = C_g^+ Q_g^+ + C_g^- Q_g^-, \quad (1)$$

where

$$Q_g^\pm = \frac{g}{16\pi^2} (\bar{s}_L \sigma^{\mu\nu} t^a d_R G_{\mu\nu}^a \pm \bar{s}_R \sigma^{\mu\nu} t^a d_L G_{\mu\nu}^a). \quad (2)$$

The chromomagnetic contribution exists in the SM (although it has not been evaluated by [3,4]) but it is expected to give a rather small contribution to  $\epsilon'/\epsilon$  since it is chirally suppressed ( $\sim \mathcal{O}(m_K^4)$ ) [21]. SUSY effects may enhance the chromomagnetic term [22] and this could be the explanation of the difference between lattice results and experiments. A large chromomagnetic term may trigger visible effects in hyperon decays [23] and in CP violating  $K \rightarrow \pi\pi\pi$  amplitudes [24]. Its coupling is constrained also by  $\epsilon$  since it may give a large contribution there, although this is rather hard to estimate. Thus for the lattice community it is certainly very interesting to compute the relevant amplitudes of the chromomagnetic operators, similarly to what has been done for the electromagnetic one, which enters the CP violating  $K_L \rightarrow \pi^0 e^+ e^-$  decay [25].

FCNC exist for different changes of flavour:

$$\mathcal{H}^{FCNC} = \mathcal{H}^{\Delta F=0} + \mathcal{H}^{\Delta F=1} + \mathcal{H}^{\Delta F=2}. \quad (3)$$

$\Delta F = 1$ , corresponding to  $\epsilon'/\epsilon$ , and  $\Delta F = 2$  processes, corresponding to  $\epsilon$  and  $B-\bar{B}$  mixing, are the most studied. Although they are of fundamental importance for the strong CP problem,  $\Delta F = 0$  FCNC processes have not received, instead, the necessary attention by the lattice community.  $\Delta F = 0$  FCNC processes are also important because from them one may derive very stringent constraints for any extension of the SM. In this framework particular important is the electric dipole moment of the neutron for which a very tight upper bound exists,  $d_N < 6.3 \times 10^{-26}$  e cm [26]. After the first studies at the end of the 80s [27], the calculation of the neutron electric dipole moment has been abandoned. This is very surprising since the renormalization of the relevant operators ( $G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a$  or  $m\bar{q}\gamma_5 q$ ) and the calculation of the necessary disconnected diagrams with stochastic sources is by now a common practice. There are several extra operators contributing to  $d_N$  that appear in extension of the SM and the matrix elements of which have never been computed [28]

$$\begin{aligned} \mathcal{H}_{eff}^{\Delta F=0} &= -\frac{i}{2} C_e \bar{q} \sigma^{\mu\nu} \gamma_5 q F_{\mu\nu} - \frac{i}{2} C_C \bar{q} \sigma^{\mu\nu} \gamma_5 t^a q G_{\mu\nu}^a \\ &- \frac{1}{6} C_g f_{abc} G_{\mu\rho}^a G_{\nu}^{b\rho} G_{\lambda\sigma}^c \epsilon_{\mu\nu\lambda\sigma}, \end{aligned} \quad (4)$$

where  $C_{e,C,g}$  are the model dependent Wilson coefficients of the effective low energy theory. After this brief discussion of FCNC amplitudes which could

be computed on the lattice with interesting implications, let us return to the results which have been presented at this Conference and to the progress which have been achieved since Lattice 2000.

### 3. General framework

Physical kaon weak decay amplitudes can be described with negligible error (of  $\mathcal{O}(\mu^2/M_W^2)$ , where  $\mu$  is the renormalization scale) in terms of matrix elements of the effective weak Hamiltonian  $\langle \pi\pi | \mathcal{H}_{eff}^{\Delta S=1} | K \rangle$ , written as combination of Wilson coefficients and renormalized local operators

$$\mathcal{H}_{eff}^{\Delta S=1} = -\frac{G_F}{\sqrt{2}} \sum_i C_i(\mu) \hat{Q}_i(\mu). \quad (5)$$

The sum is over a complete set of operators, which depend on  $\mu$ . In general there are 12 four-fermion operators and two dimension-five operators: a chromomagnetic one and an electromagnetic one. As discussed before, in the SM the contribution of the dimension-five operators is usually neglected. The calculation of the matrix elements must be done non-perturbatively, and this is the rôle of the lattice, whereas the Wilson coefficients can be computed in perturbation theory.

#### Wilson coefficients and renormalized operators

For  $\mathcal{H}_{eff}^{\Delta S=1}$  the Wilson coefficients have been computed at the next-to-leading order [29]. The perturbative calculation is reliable provided that the scale  $\mu$  is large enough,  $\mu \gg \Lambda_{QCD}$ . In this respect calculations performed below the charm quark mass ( $m_c \sim 1.3$  GeV) are, in my opinion, suspicious. In fact, either  $\mu \ll m_c$ , and then perturbation theory is questionable, or  $\mu \sim m_c$ , and then the effective three-flavour weak Hamiltonian (with propagating up, down and strange quarks) cannot be properly matched to the four-flavour theory (up, down, strange and charm) because of the presence of operators of higher dimension which contribute at  $\mathcal{O}(\mu^2/m_c^2)$ . Wilson coefficients and matrix elements of the operators  $\hat{Q}_i(\mu)$  separately depend on the choice of the renormalization scale and scheme. In the lattice approach, it is possible to show that this dependence cancel in physical quantities, up to higher-order corrections in the perturbative expansion of the Wilson coefficients.

Matching of bare (divergent) lattice operators,  $Q_i(a)$  to the renormalized ones is obtained by computing suitable renormalization constants  $Z_{ik}(\mu a)$

$$\begin{aligned} \mathcal{A}_{I=0,2}^i(\mu) &= \langle \pi\pi | \hat{Q}_i(\mu) | K \rangle_{I=0,2} \\ &= \sum_k Z_{ik}(\mu a) \langle \pi\pi | Q_k(a) | K \rangle_{I=0,2}, \end{aligned} \quad (6)$$

where  $a$  is the lattice spacing. The *ultra-violet* (UV) problem, which deals with the construction of finite matrix elements of renormalized operators constructed from the bare lattice ones, has been addressed in a series of papers [30]-[35] and is, at least in principle, completely solved. The remaining difficulties are practical ones. On the one hand, the  $\Delta I = 1/2$  operators suffer from power divergences in the ultraviolet cutoff,  $1/a$ . These divergences, that cannot be subtracted using perturbation theory [1], can be eliminated by performing suitable numerical subtractions. The subtraction procedure, however, suffers from systematic uncertainties which are difficult to keep under control, see below. On the other hand, the perturbative calculation of the logarithmically divergent (or finite)  $Z_{ik}(\mu a)$  is rather inaccurate. Several non-perturbative methods have been developed in order to compute  $Z_{ik}(\mu a)$  non-perturbatively [36]-[39] and the uncertainties vary between 1%, for the simplest bilinear operators, to 10÷25%, in the case of the four-fermion operators of interest. An accurate determination of  $Z_{ik}(\mu a)$  for the  $\Delta I = 1/2$  operators has not been achieved to date and more work is needed in this direction. Let me also recall a method for a gauge-invariant non-perturbative renormalization of the relevant operators which was proposed in [40] and never tried, in spite of the fact that it should be rather easy to implement. This proposal is based on the study, at short Euclidean distances  $|x| \ll 1/\Lambda_{QCD}$ , of suitable correlators. For example, in order to renormalize bilinear operators of the form  $J(x) = \bar{q}(x)\Gamma q(x)$ , one should compute numerically  $\langle 0 | J(x)^\dagger J(0) | 0 \rangle$  and compare the results with the expression obtained in perturbation theory in a given popular renormalization scheme ( $\overline{MS}$  or  $RI-MOM$  for example). The method avoids the problem of the window,  $a \ll |x| \ll 1/\Lambda_{QCD}$ , by successive matching of the renormalization conditions [39] and can be extended to four fermion operators. For bilinear operators, the necessary two-loop perturbative calculations, including the finite terms, have been completed and the extension to four fermion operators is underway [41].

Since numerical simulations are performed at finite values of the lattice spacing,  $a^{-1} \sim 2 \div 4$  GeV, another source of uncertainty in the determination of the matrix elements comes from discretization errors. They are of  $\mathcal{O}(\Lambda_{QCD}a)$ ,  $\mathcal{O}(|\vec{p}|a)$  or  $\mathcal{O}(m_q a)$ , where  $\vec{p}$  is a typical hadron momentum and  $m_q$  the quark mass. The simplest strategy to reduce discretization effects consists in computing physical quantities at several values of the lattice spacing and then extrapolate to  $a \rightarrow 0$ . A different approach is to reduce

discretization errors from  $\mathcal{O}(a)$  to  $\mathcal{O}(a^2)$  by improving the lattice action and operators [39]. A systematic study of the improvement of four fermion operators is still to be done. Note that with DWF [42] or overlap Fermions [43] the errors are automatically of  $\mathcal{O}(a^2)$  [44]. Discretization errors correspond to the matching problems in effective theories with a low cutoff recently discussed by Cirigliano, Donoghue and Golowich [46]. In this respect the problem is softer for the lattice approach since numerical simulations are already performed at relatively large scales.

#### $K \rightarrow \pi$ and $K \rightarrow \pi\pi$ matrix elements

Two main roads have been suggested in the past in order to obtain  $\mathcal{A}_{I=0,2}^i(\mu)$ :

i) Compute the  $K \rightarrow 0$  and  $K \rightarrow \pi$  matrix elements  $\langle 0|\hat{Q}_i(\mu)|K\rangle$  and  $\langle \pi|\hat{Q}_i(\mu)|K\rangle$  and then derive  $\langle \pi\pi|\hat{Q}_i(\mu)|K\rangle_{I=0,2}$  using soft pion theorems [30,31]. In this case the  $K \rightarrow \pi\pi$  amplitudes can be evaluated only at the lowest order of the chiral expansion and FSI effects in the physical amplitudes are lost. This problem poses serious difficulties to lattice calculations based on the extraction of the  $K \rightarrow \pi\pi$  amplitudes from the  $K \rightarrow \pi$  matrix elements using chiral perturbation theory ( $\chi$ PT), as done by the CP-PACS [3] and RBC Collaborations [4]. The importance of FSI effects for  $\Delta I = 1/2$  transitions was first noticed in refs. [47,48]. More recently it has been emphasized by Bertolini, Eeg and Fabbrihesi [21] and Pallante and Pich [49] that they may have large effects also for  $\epsilon'/\epsilon$ . Although the quantitative evaluation of FSI effects is still controversial [50], there is a general consensus that, for a reliable calculation of kaon decay amplitudes, it is necessary to have a good theoretical control of FSI.

ii) Compute directly  $\langle \pi\pi|\hat{Q}_i(\mu)|K\rangle_{I=0,2}$  [14,15,32,33]. The main difficulty in the latter case is due to the relation between  $K \rightarrow \pi\pi$  matrix elements computed in a finite Euclidean space-time volume and the corresponding physical amplitudes (the *infrared* problem). The infrared problem arises from two sources: i) the unavoidable continuation of the theory to Euclidean space-time and ii) the use of a finite volume in numerical simulations. An important progress has been achieved by Lellouch and Lüscher (LL), who derived a relation between the lattice  $K \rightarrow \pi\pi$  matrix elements in a finite volume and the physical kaon-decay amplitudes [14]. An alternative discussion of boundary effects and the LL-formula, based on a study of the properties of correlators of local operators was presented in ref. [15]. In this approach the LL-formula is derived for all elastic states under the inelastic threshold, with exponen-

tial accuracy in the quantization volume. As a consequence the relation between finite-volume matrix elements and physical amplitudes, derived by LL for the lowest seven energy eigenstates, can be extended to all elastic states under the inelastic threshold. It can also be explicitly demonstrated how finite volume correlators converge to the corresponding ones in infinite volume. The derivation of [15] is based on the property of correlators of local operators which can be expressed, with exponential accuracy, both as a sum or as an integral over intermediate states, when considering volumes larger than the interaction radius and Euclidean times  $0 < t \simeq L$ . It can also be shown that it is possible to extract  $K \rightarrow \pi\pi$  amplitudes also when the kaon mass,  $m_K$ , does not match the two-pion energy, namely when the inserted operator carries a non-zero energy-momentum. Such amplitudes are very useful, for example, in the determination of the coefficients of operators appearing at higher orders in  $\chi$ PT, as illustrated by the numerical results for  $\Delta I = 3/2$  transitions presented in sec. 4.

Let us sketch now the derivation of the result with an illustrative example which is not explicitly written in [15]. Consider the following Euclidean T-products (correlators):

$$\begin{aligned} G(t, t_K) &= \langle 0|T[J(t)\hat{Q}_i(0)K^\dagger(t_K)]|0\rangle, \\ G(t) &= \langle 0|T[J(t)J(0)]|0\rangle, \\ G(t_K) &= \langle 0|T[K(t_K)K^\dagger(0)]|0\rangle, \end{aligned} \quad (7)$$

where  $J$  is a scalar operator which excites (annihilates) zero angular momentum  $\pi\pi$  states from (to) the vacuum and  $K^\dagger$  is a pseudoscalar source which excites a kaon from the vacuum ( $t > 0$  and  $t_K < 0$ ). At large time distances we have

$$\begin{aligned} G(t, t_K) &\rightarrow \\ V \sum_n &\langle 0|J|\pi\pi(n)\rangle_V \langle \pi\pi(n)|\hat{Q}_i|K\rangle_V \langle K|K^\dagger|0\rangle_V \\ &\exp(-W_n t - m_K t_K), \end{aligned} \quad (8)$$

$$G(t) \rightarrow V \sum_n \langle 0|J|\pi\pi(n)\rangle_V \langle \pi\pi(n)|J|0\rangle_V \exp(-W_n t)$$

$$G(t_K) \rightarrow V \langle 0|K|K\rangle_V \langle K|K^\dagger|0\rangle_V \exp(-m_K t_K)$$

From a study of the time dependence of  $G(t, t_K)$ ,  $G(t)$  and  $G(t_K)$  we may extract: 1) the kaon mass  $m_K$ ; 2) the two-pion energies on the finite lattice volume,  $W_n$ . As shown by M. Lüscher in [57], the  $W_n$  are related to the infinite volume phase-shift of the two pions,  $\delta(k)$ , via the following relations

$$W_n = 2\sqrt{m_\pi^2 + k^2}, \quad \frac{\phi(q) + \delta(k)}{\pi} = n,$$

$$n = 1, 2, \dots, \quad q = \frac{kL}{2\pi}, \quad (9)$$

where  $n$  is a non-negative integer<sup>1</sup> and the function  $\phi(q)$  is defined in [57]. 3) the operator matrix elements on a finite volume  $\langle \pi\pi(n) | \hat{Q}_i | K \rangle_V$ ,  $\langle \pi\pi(n) | J | 0 \rangle_V$  and  $\langle K | K^\dagger | 0 \rangle_V$ . Moreover by a suitable choice of the lattice parameters it is possible to match one of the two-pion energies in such a way that  $m_K = W_{n^*}$ . In practice, since it will be possible to disentangle only the first few states, the matrix elements will be computed with  $n^* = 0 \div 2$ .

The fundamental point is that it is possible to relate the finite-volume Euclidean matrix element  $\langle \pi\pi(n^*) | \hat{Q}_i | K \rangle_V$  with the absolute value of the physical amplitude  $\langle \pi\pi | \hat{Q}_i | K \rangle$ :

$$\begin{aligned} \langle \pi\pi | \hat{Q}_i | K \rangle &= \sqrt{\mathcal{F}} \langle \pi\pi(n^*) | \hat{Q}_i | K \rangle_V, \\ \mathcal{F} &= 32\pi^2 V^2 \frac{\rho_V(E) E m_K}{\kappa(E)}, \end{aligned} \quad (10)$$

where

$$\begin{aligned} \kappa(E) &= \sqrt{\frac{E^2}{4} - m_\pi^2}, \\ \rho_V(E) &= \frac{dn}{dE} = \frac{q\phi'(q) + k\delta'(k)}{4\pi k^2} E. \end{aligned} \quad (11)$$

The last expression in eq. (11) is the one which one would heuristically derive from a naïve interpretation of  $\rho_V(E)$  as the density of states, cf. eq. (9). There are however, some technical subtleties with such an interpretation which will not be discussed here. Eq. (10) holds also when  $m_K \neq W_n$  and the operator carries non-zero energy-momentum [15]. By varying the kaon and pion masses and momenta, one may then fit the coefficients of the chiral expansion of  $\langle \pi\pi | \hat{Q}_i | K \rangle$  and use these coefficients to extrapolate to the physical point. This procedure is necessary, at present, since it is not possible to compute the matrix elements at realistic values of the quark masses and in the unquenched case. To give an explicit example, the matrix elements of  $\hat{Q}_{7,8}$  (for generic meson masses, one pion at rest and the other with a given momentum corresponding to an energy  $E_\pi$ ) can be written as [7,45]

$$\begin{aligned} -i\mathcal{M}_{7,8}(m_K, m_\pi, E_\pi) &= \gamma^{7,8} \\ &+ \delta_1^{7,8} \left( \frac{m_K(m_\pi + E_\pi)}{2} - m_\pi E_\pi \right) + \dots \end{aligned} \quad (12)$$

A fit to the lattice data for  $\mathcal{M}_{7,8}$ , extracted from suitable correlation functions, allows the determination of the couplings  $\gamma^{7,8}$ ,  $\delta_1^{7,8}$ , etc. The results of

<sup>1</sup>For  $n = 0$ , there are two solutions: one corresponding to  $k = 0$  which is spurious, the other giving the Lüscher relation between the finite volume energy and the scattering length.

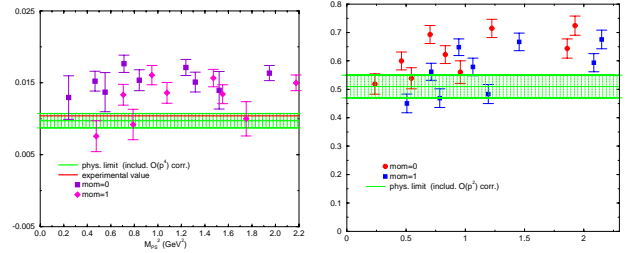


Figure 1. a. Chiral behaviour of the matrix element of  $Q_4$ ,  $\mathcal{M}_4(m_K, m_\pi, E_\pi)$ . The extrapolation to the physical point using only the data corresponding to the lightest masses, without the effect of the chiral logarithms and using the operators of ref. [12], is also shown as a shadowed band. The experimental result, computed as explained in the text, is also given as a line. b. Chiral behaviour of the matrix element of  $Q_8$ ,  $\mathcal{M}_8(m_K, m_\pi, E_\pi)$ . The extrapolation to the physical point using only the data corresponding to the lightest masses, without the effect of the chiral logarithms and using the operators of ref. [45], is also shown as a shadowed band.

the extrapolated amplitudes, if chiral logarithms are included, are independent of the cutoff used in the chiral theory.

#### 4. Numerical Results

In this section, the latest numerical results from lattice QCD for  $K \rightarrow \pi\pi$  and  $K \rightarrow \pi$  amplitudes and for  $\hat{B}_K$  are reviewed, together with a comparison with other approaches.

##### $\Delta I = 3/2$ transitions and $K^0 - \bar{K}^0$ mixing

The SPQCDR Collaboration [7,8] has presented (quenched) results for both  $\Delta I = 3/2$  and  $\Delta I = 1/2$  amplitudes, obtained from a calculation of  $K \rightarrow \pi\pi$  matrix elements following the strategy of refs. [14, 15]. For  $\Delta I = 3/2$  transitions, an extended study of the chiral behaviour of the matrix elements of the operators  $\hat{Q}_{4,7,8}$ , renormalized non-perturbatively, was performed. In figs. 1a and 1b,  $\mathcal{M}_4(m_K, m_\pi, E_\pi)$  and  $\mathcal{M}_8(m_K, m_\pi, E_\pi)$  are shown as a function of the kaon mass. Note that the amplitude does depend on three independent quantities that are let to vary, namely  $m_K$ ,  $m_\pi$  and  $E_\pi$  (one of the two pions is always at rest). In the same figures the extrapolation to the physical point, performed by using the chiral expansion of the matrix elements as in eq. (12), is given as a band. In fig. 1a the experimental number, extracted from the  $K^+ \rightarrow \pi^+\pi^0$  decay rate using the Wilson coefficient of  $\hat{Q}_4$  computed at the NLO, is also shown. The extrapolations are preliminary, since they do not include the effects of (quenched) chiral logarithms that have not been computed yet for the kinemat-

ical configurations used in this study. For some of the points, masses and momenta are too large to use chiral perturbation theory, and they have not been included in the fit. The preliminary results in fig. 1a already give us an interesting physics information. In the chiral limit, and using  $SU(3)$  symmetry, one may relate the  $K^+ \rightarrow \pi^+\pi^0$  amplitude to the  $K^0-\bar{K}^0$  mixing parameter  $\hat{B}_K$  [58]. In this limit, a large value for  $\hat{B}_K$ , as found in lattice calculations and unitarity triangle analyses [59],  $\hat{B}_K \sim 0.85$ , would lead to a value of the  $K^+ \rightarrow \pi^+\pi^0$  amplitude larger than the experimental one by  $\sim 50\%$ . The extrapolation of the results of fig. 1a, together with those for  $\hat{B}_K$  obtained by the same collaboration [9] given in table 4, demonstrate that chiral corrections to both the  $K^+ \rightarrow \pi^+\pi^0$  amplitude and  $\hat{B}_K$  can easily reconcile a large value of  $\hat{B}_K$  with the experimental  $K^+ \rightarrow \pi^+\pi^0$  amplitude.

It is very instructive to compare the results for  $\hat{Q}_7$  and  $\hat{Q}_8$  with those obtained in other approaches, table 3. In the case of  $\hat{Q}_8$  we notice the nice agreement between lattice results obtained from  $K \rightarrow \pi$  matrix elements using chiral perturbation theory and those from the first direct calculation of  $K \rightarrow \pi\pi$  amplitudes. Similar numbers were obtained by the CP-PACS [3] and RBC [4] collaborations, which computed  $K \rightarrow \pi$  matrix elements with DWF. These groups presented the results in different renormalization schemes and scales and for this reason the values have not been included in the table. There is, however, a very large discrepancy of the lattice results with those obtained with dispersive methods [60] or with the  $1/N_c$  expansion [61], whose results would correspond to a huge value of the  $B$ -parameter,  $B_8 = 3 \div 7^2$ . In order to reproduce the experimental results for  $\epsilon'/\epsilon$  within the Standard Model, such a large value of  $B_8$  implies a stratospheric value for  $B_6$ . After Kaon 1999 a new analysis, performed with spectral function techniques by Bijmns, Gamiz and Prades appeared [63]. In this paper, a value of  $\langle \hat{Q}_8 \rangle$  much lower than in refs. [60,61] was found, although still about a factor of two larger than lattice determinations. The very low value of  $\langle \hat{Q}_7 \rangle$  found from the lattice  $K \rightarrow \pi\pi$  calculation originates from large cancellations occurring in the renormalization of the relevant operator and requires further investigation.

Lattice predictions for  $\hat{B}_K$  have been very stable over the years, with a central value centered around 0.85. This is essentially also the value extracted from Unitarity Triangle Analyses [59],

<sup>2</sup>Other results for  $\hat{Q}_7$  and  $\hat{Q}_8$  can be found in refs. [21] and [62].

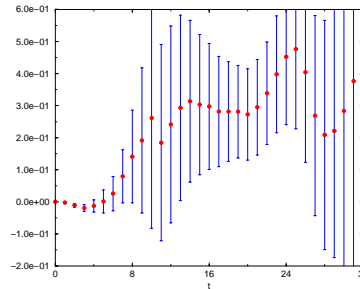


Figure 2. First signal for the matrix element  $\langle \pi\pi | Q^- | K \rangle$  at the matching point  $m_K \sim W_0$ , obtained with 400 configurations and  $V = 24^3 \times 64$ .

thus confirming that lattice QCD can predict (and not only postdict) physical quantities. In table 4 the lattice world average is given, together with some of the latest lattice results.  $\hat{B}_K$  from CP-PACS [64] (with operators renormalized perturbatively) and RBC [53] (with operators renormalized non-perturbatively) has been obtained with DWF. This formulation of QCD should guarantee a better control of the chiral behaviour of the regularized theory with respect to Wilson-like fermions. The results from ref. [9] have been obtained using the non-perturbatively improved Wilson-like action, with a new method based on the Ward Identities [65], which save us from the painful subtractions of the wrong chirality operators, which was necessary in the past. Similar strategies have been pursued with twisted-mass fermions [38]. The results with DWF are about 15% below the world average and show a marked decrease at small quark masses. This could reconcile the large value of  $\hat{B}_K$  at the physical kaon mass with the low value of this parameter obtained in the chiral limit by ref. [66]. Whether the decrease at small quark masses is a physical effect, or is due to lattice artefacts (finite volume, residual chiral symmetry breaking etc.) remains to be investigated. New calculations of  $\langle \bar{K}^0 | \hat{Q}_i | K^0 \rangle$  for all possible operators  $Q_i$  which can mediate  $K^0-\bar{K}^0$  mixing in extensions of the Standard Model have been presented [9]. These results are very useful to put severe constraints on FCNC parameters of SUSY models [55].

#### $\Delta I = 1/2$ transitions and $\epsilon'/\epsilon$

For  $\Delta I = 1/2$  transitions and  $\epsilon'/\epsilon$ , the subtractions of the power divergencies, necessary to obtain finite matrix elements, are the major obstacle in lattice calculations. These divergencies arise from penguin contractions of the relevant operators, which induce a mixing with operators of lower dimensions, namely  $\bar{s}\sigma^{\mu\nu}G_{\mu\nu}^A t^A d$ ,  $\bar{s}\sigma^{\mu\nu}\gamma_5 G_{\mu\nu}^A t^A d$ ,  $\bar{s}d$  and  $\bar{s}\gamma_5 d$  [31]. Power divergencies and subtractions



Table 3

$K \rightarrow \pi\pi$  matrix elements in  $\text{GeV}^3$ , at  $\mu = 2 \text{ GeV}$  in the  $\overline{MS}$  scheme, from lattice calculations (first three rows) and other approaches.

Quantity	Experiments	Theory	
Reference	Method	$\langle \hat{Q}_8 \rangle$	$\langle \hat{Q}_7 \rangle$
SPQ <sub>CDR</sub> [8] 2001	$K \rightarrow \pi\pi$	$0.53 \pm 0.06$	$0.02 \pm 0.01$
SPQ <sub>CDR</sub> [9] 2001	$K \rightarrow \pi + \chi\text{PT}$	$0.49 \pm 0.06$	$0.10 \pm 0.03$
APE [55] 1999	$K \rightarrow \pi + \chi\text{PT}$	$0.50 \pm 0.10$	$0.11 \pm 0.04$
Amherst [60] 1999	Dispersion relations	$2.22 \pm 0.67$	$0.16 \pm 0.10$
Marseille [61] 1998	$1/N_c$	$3.50 \pm 1.10$	$0.11 \pm 0.03$
BGP [63] 2001	Spectral functions	$1.2 \pm 0.5$	$0.26 \pm 0.03$

Table 4

$B_K$  in the  $\overline{MS}$  scheme at the renormalization scale  $\mu = 2 \text{ GeV}$  and the renormalization group invariant  $\hat{B}_K$  from recent lattice calculations. The results from SPQ<sub>CDR</sub>, CP-PACS and RBC Collaborations are quenched.

Quantity	Experiments	Theory	
Reference	Method	$B_K^{\overline{MS}}(2 \text{ GeV})$	$\hat{B}_K$
World Average		$0.63 \pm 0.03 \pm 0.10$	$0.86 \pm 0.06 \pm 14$
SPQ <sub>CDR</sub> [9] 2001	with subtractions	$0.71 \pm 0.13$	$0.91 \pm 0.17$
SPQ <sub>CDR</sub> [9] 2001	Ward identity method	$0.70 \pm 0.10$	$0.90 \pm 0.13$
CP-PACS [64] 2001	DWF Pert. Ren.	$0.5746(61)(191)$	$0.787 \pm 0.008$
RBC [53] 2000	DWF Nonpert. Ren.	$0.538 \pm 0.08$	$0.737 \pm 0.011$
Ciuchini et al. [59] 2001	Unitarity Triangle	$0.70^{+0.23}_{-0.11}$	$0.90^{+0.30}_{-0.14}$

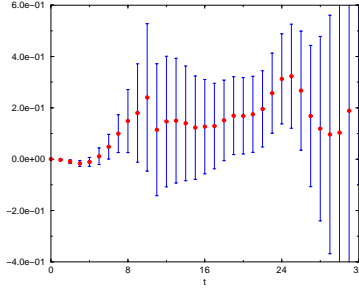


Figure 3. First signal for the matrix element  $\langle \pi\pi | Q_6 | K \rangle$  at the matching point  $m_K \sim W_0$ , obtained with 400 configurations and  $V = 24^3 \times 64$ .

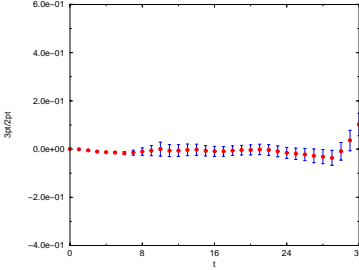


Figure 4.  $\langle \pi\pi | \bar{s}\gamma_5 d | K \rangle$  at the matching point  $m_K \sim W_0$ , obtained with 400 configurations and  $V = 24^3 \times 64$ . At the matching point  $\langle \pi\pi | \bar{s}\gamma_5 d | K \rangle$  decreases by a huge factor and thus the subtraction for  $\langle \pi\pi | Q_6 | K \rangle$  is very small.

are encountered with both the strategies ( $K \rightarrow \pi\pi$  or  $K \rightarrow \pi$ ) adopted to compute the physical amplitudes. For example, with a propagating charm quark,  $Re \mathcal{A}_0$  is computed in terms of the matrix elements of  $Q^\pm$  ( $Q_1$  and  $Q_2$ ) only. In this case, due to the GIM mechanism, the subtraction is implicit in the difference of penguin diagrams with a charm and an up quark propagating in the loop:

$$Q^\pm = \bar{s}\gamma_\mu(1 - \gamma_5)u \bar{u}\gamma^\mu(1 - \gamma_5)d \pm \bar{s}\gamma_\mu(1 - \gamma_5)d \bar{u}\gamma^\mu(1 - \gamma_5)u - (u \rightarrow c). \quad (13)$$

In the past any attempt to compute  $\Delta I = 1/2$   $K \rightarrow \pi\pi$  matrix elements failed because no visible signal was observed after the subtraction of the power divergencies [67,68]. These calculations, performed about twelve years ago with very modest computer resources, on small lattices and with little statistics, were abandoned after the publication of the Maiani-Testa no-go theorem [51].

This year, the SPQ<sub>CDR</sub> Collaboration has found the first signals for both the matrix elements of  $Q^-$  and  $Q_6$ , figs. 2, 3 and 4. The difference with respect to previous attempts is given by a much larger volume and statistics, about 400 configurations with  $V = 24^3 \times 64$ , compared to a few tens of configurations on lattices with  $V = 16^3 \times 32$  (8 configurations on  $V = 24^3 \times 40$ ) [67] or 110 configuration of a lattice

with  $V = 16 \times 12 \times 10 \times 32$  (sic !!) [68]. A crucial ingredient is also to work at the point corresponding to the matching condition  $m_K = W_0$ , where  $W_0$  is the energy of the two pions at rest on the finite volume used in the simulation, namely with non degenerate strange and light quark masses<sup>3</sup>. Past calculations were always performed, instead, at the degenerate point,  $m_K = m_\pi$ . The results are very preliminary and in particular the two-pion energy in the finite volume for this channel gives a scattering length,  $a_{I=0}$ , in disagreement with expectations both in value and in the mass dependence. This is to be contrasted with the  $\Delta I = 3/2$  case, where the analysis of the scattering length  $a_{I=2}$  is in good agreement with expectations [8]. Much more work is needed to clarify these problems before trying to extrapolate the amplitude to the physical point.

At the matching point  $m_K = W_0$ , there are also results for  $\langle \pi\pi|Q_6|K \rangle$ . In this case there is no GIM mechanism at work, and a finite subtraction of the matrix element of the pseudoscalar density operator  $Q_P = (m_s - m_d)\bar{s}\gamma_5 d$  must be done (for parity, instead, the subtraction of the scalar operator  $Q_S = (m_s + m_d)\bar{s}d$  is not necessary). The coefficient of the mixing of  $Q_P$  with  $Q_6$  is quadratically divergent<sup>4</sup>

$$\hat{Q}_6 \sim Q_6 + \frac{C_P(\alpha_s)}{a^2} Q_P \quad (14)$$

If not for the explicit chiral symmetry breaking of the lattice action,  $\langle \pi\pi|Q_P|K \rangle$  would vanish by the equation of motions when  $m_K = W_{n^*}$ . With an improved action,  $\langle \pi\pi|Q_P|K \rangle$  is of  $\mathcal{O}(a^2)$  and thus a finite subtraction is necessary. The preliminary results shown in fig. 4 show that the value of  $\langle \pi\pi|\bar{s}\gamma_5 d|K \rangle$  drops by a factor of about 20 with respect to  $\langle \pi\pi|Q_6|K \rangle$  when  $m_K = W_0$  (it is of the same order when the kinematics is not matched) and thus the subtraction is not very critical. The signal itself is very noisy however, see fig. 3, and even with 400 configurations the statistical error on  $\langle \pi\pi|Q_6|K \rangle$  is about 50%. Thus a rather large statistics will be necessary to obtain a relatively accurate result.

$K \rightarrow \pi\pi$  amplitudes in the chiral limit can also be extracted from the calculation of  $K \rightarrow \pi$  matrix elements, and indeed this has been the most popular method in lattice calculations of  $\Delta I = 3/2$  transitions. The main advantage of this approach is

that the three-point correlators, necessary for the extraction of the matrix elements, are less noisy than the four-point correlators used in the  $K \rightarrow \pi\pi$  case. Moreover, for  $K \rightarrow \pi$  matrix elements, finite volume corrections are exponentially small as  $L \rightarrow \infty$ . The main disadvantage is that the  $K \rightarrow \pi\pi$  amplitudes are obtained, using soft pion theorems, in the chiral limit only and then the effect of FSI is definitively lost. If these are important for  $\Delta I = 1/2$  channels there is no hope, then, to recover the physical amplitudes. In order to make the  $K \rightarrow \pi$  matrix element finite at least a subtraction is necessary, even in the absence of explicit chiral symmetry breaking in the action. With Wilson-like Fermions, for which chiral symmetry is explicitly broken, the number of subtractions makes this approach unpracticable as also demonstrated by the failure of past attempts [67,68]. The method has acquired a renewed popularity, instead, with recent formulations of the lattice theory, DWF or overlap Fermions, for which chiral symmetry breaking is absent or strongly reduced in practice. In this case, and under the hypothesis that chiral symmetry is under control, a finite amplitude can be obtained by subtracting the scalar density amplitude with a suitable coefficient  $C_i$  which depends on the operator at hand

$$\langle \pi|Q_i^{sub}|K \rangle = \langle \pi|Q_i|K \rangle - C_i \langle \pi|Q_S|K \rangle. \quad (15)$$

The power divergent coefficients  $C_i$  cannot be computed perturbatively. Following ref. [30], when chiral symmetry is preserved, the  $C_i$  can be determined by the condition that the  $K \rightarrow 0$  matrix element of the subtracted operator vanishes

$$\langle 0|Q_i - C_i Q_P|K \rangle = 0. \quad (16)$$

The coefficients  $C_i$  have been obtained either using non degenerate quarks,  $m_s \neq m_d$ , by the RBC Collaboration, or from the derivatives of the 2-point correlation function with respect to the quark mass, by the CP-PACS Collaboration. Numerically, the matrix element of the subtracted operator is much smaller than the unsubtracted one, as shown by fig. 5, taken from the recent work of the RBC collaboration [4]. This implies that any systematic uncertainty which enters in the subtraction procedure can have huge effects in the determination of the physical amplitudes. After the subtraction, the chiral dependence of the matrix element has to be fitted in order to extrapolate it to the chiral limit. Both groups have included the logarithmic corrections which arise in quenched  $\chi$ PT in the fit and, in some cases, polynomial corrections of higher order in  $m_\pi^2$ . The chiral behaviour observed by CP-PACS is very satisfactory,

<sup>3</sup>In this case, as in the case of degenerate quark masses, the subtraction of operators of lower dimensions is not necessary [33].

<sup>4</sup>The mixing with the chromomagnetic operator is not discussed here. This mixing is a small, finite correction which can be ignored to simplify the discussion.



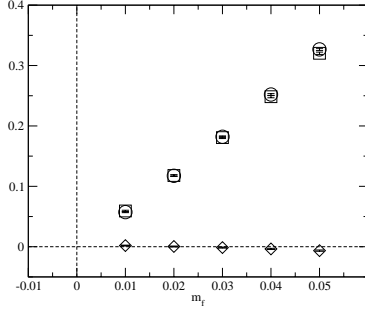


Figure 5.  $\langle \pi | Q | K \rangle$  for the unsubtracted operator,  $Q_6$  (squares), the subtraction,  $C_6 Q_S$  (Circles), and the total,  $Q_i^{sub}$  see eq. (15), as a function of the quark mass  $m_f$ . The results are from the RBC Collaboration.

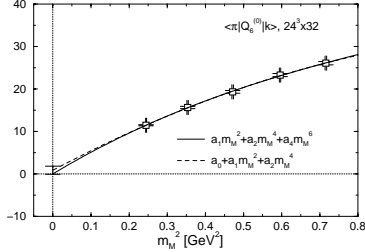


Figure 6. Chiral behaviour of the matrix element  $\langle \pi | Q_6 | K \rangle$  from the CP-PACS Collaboration. The curves represent different fits used to extrapolate to the chiral limit.

as shown by fig. 6, less good in the case of the RBC Collaboration. The difference may be due to the fact that a different gluon action is used in the two cases, corresponding to smaller chirally breaking effects for CP-PACS. This can be monitored by measuring the so called “residual mass” which should be zero for perfect chirality and is about a factor of ten smaller for CP-PACS than for RBC. For a more extended discussion on this important point the reader can refer to [3,4]. Let me also mention that the overall renormalization constants have been computed perturbatively by CP-PACS [34] and non-perturbatively by RBC [37]. My compendium of the physics results obtained by the two groups is given in table 2. A few observations are necessary at this point. First of all,  $ReA_0$ , and consequently  $ReA_0/ReA_2$ , from the RBC Collaboration are in good agreement with experimental values, contrary to CP-PACS which finds  $ReA_0$  smaller by about a factor of two than the experimental number. Since the two groups use the same lattice formulation of the theory and differ only by, hopefully, marginal details, the reason of this difference should be clarified. In both cases, however,  $\epsilon'/\epsilon$  is in total disagreement with the data. The main reason is that the value of the matrix element of  $Q_6$  is much smaller than what would be necessary to repro-

duce the experimental value (it corresponds approximately to  $B_6 = 0.3 \div 0.4$ ). Let me list a number of sources of systematic errors which may explain these embarrassing results:

1) Chirality By working with DWF at a finite fifth dimension,  $N_5 = 16$ , a residual chiral symmetry breaking is present in the theory. The amount of residual symmetry breaking is parametrized by a mass scale denoted by  $m_{res} \sim 0.2 \div 2.0$  MeV. In the presence of explicit chiral symmetry breaking, the coefficients  $C_i$  determined from  $K \rightarrow 0$  differs from the correct one and this may induce an error of  $\mathcal{O}(m_{res} a^{-2}) \sim (200 \text{ MeV})^3$  on matrix elements which are of the order of  $\Lambda_{QCD}^3 \sim (300 \text{ MeV})^3$ . Both groups claim to have this point fully under control [3,4]. A calculation at a larger value of  $N_5$ , with all the other parameters unchanged, would be very useful to clarify the situation.

2) Matching below the charm mass Both groups have so far presented results at a renormalization scale just below the charm quark mass. The matching of the effective theory is rather problematic at such low scales.

3) Extra Quenched Chiral Logarithms As shown by fig. 5, the subtraction of the power divergencies is very critical. Besides the effects discussed in 1), another delicate problem has been recently raised by Golterman and Pallante [13]. In the quenched theory the  $(8, 1)$  operators, such as  $Q^-$  and  $Q_6$ , do not belong anymore to irreducible representations of the chiral group and this gives rise to spurious chiral logarithms. These logarithms affect the subtraction procedure and have not been taken into account in the analyses of RBC and CP-PACS. A preliminary numerical study to try to get rid of these effect has been discussed at this conference [69]

4) Higher order chiral corrections and FSI Higher order chiral contributions, among which FSI have also to be accounted, can produce large variations of the matrix elements between the chiral limit and the physical point. In simulations where only  $K \rightarrow \pi$  matrix elements are computed,  $K \rightarrow \pi\pi$  amplitudes can only be obtained at lowest order in  $\chi$ PT where these physical effects are missing.

5) New Physics As noticed by Murayama and Masiero [22], it is possible to produce large effects for  $\epsilon'/\epsilon$  in SUSY without violating other bounds coming from FCNC. If the lattice results are correct, this is an open possibility. Before invoking new physics, at least the result for  $ReA_0$  should be established with more confidence. Given that the two groups still don't agree on this quantity, more work is needed.

## 5. Conclusion and outlook

In this review, because of space limitations, I discussed only a subset of the results for light-quark matrix elements presented at Lattice 2001. Although the focus has been on kaon weak matrix elements, the general discussion relative to renormalization, improvement and calculation of physical amplitudes applies to many other cases as well. A renewed activity in lattice calculations of kaon decays and mixing has developed in the last two years. For  $\Delta I = 3/2$  transitions,  $K^+ \rightarrow \pi^+\pi^0$  and electroweak penguin amplitudes, new and more precise results have been obtained. In this case, removal of lattice artefacts, by extrapolation to the continuum and/or improvement, and unquenched calculations are around the corner. For  $\Delta I = 1/2$  transitions, it has been shown that direct computations of  $K \rightarrow \pi\pi$  amplitudes, including FSI, is possible [14,15], although the practical implementation with sufficient accuracy will require more time. First results, obtained by using  $K \rightarrow \pi$  lattice matrix elements computed with DWF and soft-pion theorems, show a striking disagreement with the experiments for  $\epsilon'/\epsilon$ . For  $Re\mathcal{A}_0$ , there is a factor of two between the values found by the CP-PACS [3] and RBC [4] Collaborations. More work is needed in this case to clarify all these points. Many related physical quantities, like the strong interaction phase-shifts for  $\pi\pi$  scattering, the chromomagnetic operator matrix elements, semileptonic  $K \rightarrow \pi\pi\nu_l$  amplitudes and the scalar semileptonic form factor will also be obtained as a byproduct from lattice investigations of non-leptonic kaon decays. In particular, the study of the  $K \rightarrow \pi\pi\nu_l$  amplitude is very useful since it contains all the ingredients relative to FSI without having the extra difficulties due to the power divergences of the non-leptonic amplitudes, and the corresponding necessary subtractions.

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## REFERENCES

1. L. Lellouch, *Nucl. Phys. (Proc. Suppl.)* **94** (2001) 142.
2. S. Sint, *Nucl. Phys. (Proc. Suppl.)* **94** (2001) 79. [arXiv:hep-lat/0011081].
3. S. Aoki *et al.*, CP-PACS Collaboration, presented by J. Noaki at this Conference, [CP-PACS Collaboration], arXiv:hep-lat/0110142; see also hep-lat/0108013.
4. T. Blum *et al.*, from the presentations of the RIKEN-BNL-Columbia (RBC) Collaboration, by T. Blum, C. Cristian and R. Mawhinney, arXiv:hep-lat/0110185 and arXiv:hep-lat/0110144; see also arXiv:hep-lat/0110075.
5. M. Beneke plenary talk given at this Conference.
6. J. Donoghue, talk presented at Kaon 2001, Pisa (Italy), June 12–17 2001.
7. P. Boucaud *et al.*, the SPQCDR Collaboration, presented by D. Lin at this Conference, arXiv:hep-lat/0110206.
8. P. Boucaud *et al.*, the SPQCDR Collaboration, presented by M. Papinutto at this Conference, arXiv:hep-lat/0110169.
9. D. Becirevic *et al.*, the SPQCDR Collaboration, presented by D. Becirevic at this Conference.
10. C. W. Bernard and M. F. Golterman, *Phys. Rev. D* **53** (1996) 476.
11. M. F. Golterman and K. C. Leung, *Phys. Rev. D* **56** (1997) 2950; *Phys. Rev. D* **57** (1998) 5703; *Phys. Rev. D* **58** (1998) 097503, hep-lat/9805032.
12. M. Golterman and E. Pallante, *JHEP* **0008** (2000) 023.
13. M. Golterman and E. Pallante, hep-lat/0108010; also E. Pallante at this Conference, arXiv:hep-lat/0110183.
14. L. Lellouch and M. Lüscher, *Commun. Math. Phys.* **219** (2001) 31.
15. C. J. Lin *et al.*, hep-lat/0104006.
16. Review of Particle Physics, *Eur. Phys. J. C* **15** (2000) 1.
17. A. Lai *et al.*, the NA48 Collaboration, arXiv:hep-ex/0110019; I. Mikulec *et al.*, the NA48 Collaboration, talk presented at Kaon 2001, Pisa (Italy), June 12–17 2001.
18. A. Glazov *et al.*, the KTeV Collaboration, talk presented at Kaon 2001, Pisa (Italy), June 12–17 2001.
19. C. Greub, arXiv:hep-ph/9911348.
20. G. Isidori, arXiv:hep-ph/0110255.
21. S. Bertolini *et al.*, *Nucl. Phys. B* **514** (1998) 93; *Phys. Rev. D* **63** (2001).
22. A. Masiero and H. Murayama, *Phys. Rev. Lett.* **83** (1999) 907.
23. X. G. He *et al.*, *Phys. Rev. D* **61** (2000) 071701.
24. G. D'Ambrosio *et al.* *Phys. Lett. B* **480** (2000)

- 164.
25. D. Becirevic *et al.*, *Phys. Lett.* **B501** (2001) 98.
26. E. Hinds, talk presented at Kaon 2001, Pisa (Italy), June 12–17 2001.
27. S. Aoki *et al.* *Phys. Rev. Lett.* **65** (1990) 1092.
28. S. Pokorski *et al.*, *Nucl. Phys. B* **570** (2000) 81.
29. A. J. Buras, arXiv:hep-ph/0109197 and refs. therein.
30. C. W. Bernard *et al.*, *Phys. Rev.* **D32** (1985).
31. M. Bochicchio *et al.*, *Nucl. Phys.* **B262** (:) 1988  
L. Maiani *et al.*, *Phys. Lett.* **176B** (1986) 445  
and *Nucl. Phys.* **B289** (1987) 505.
32. C. Bernard *et al.*, *Nucl. Phys. (Proc. Suppl.)* **4**  
1988 483.
33. C. Dawson *et al.*, *Nucl. Phys.* **B514** (1998) 313.
34. S. Aoki *et al.*, *Phys. Rev.* **D59** (1999) 094505;  
*Phys. Rev. D60* (1999) 114504; *Phys. Rev.* **D63**  
(2001) 054504 and in preparation.
35. S. Capitani and L. Giusti, hep-lat/0011070.
36. G. Martinelli *et al.*, *Nucl. Phys.* **B445** (1995) 81;  
A. Donini *et al.*, *Eur. Phys. J. C* **10** (1999) 121.
37. RBC Collaboration, C. Dawson *et al.* *Nucl. Phys.*  
(*Proc. Suppl.*) **94** (2001) 613
38. M. Guagnelli *et al.*, presented by C. Pena at this  
Conference, arXiv:hep-lat/0110097.
39. For a review, see M. Luscher, Les Houches Summer  
School in Theoretical Physics, Les Houches,  
France, 28 Jul - 5 Sep 1997, hep-lat/9802029 and  
references therein.
40. G. Martinelli *et al.*, *Phys. Lett.* **B411** (1997)  
41; G. Martinelli, *Nucl. Phys. (Proc. Suppl.)* **73**  
(1999) 58.
41. F. Del Bello and G. Martinelli, work in progress.
42. D.B. Kaplan, *Phys. Lett.* **B288** (1992) 342, *Nucl.*  
*Phys.(Proc. Suppl.)* **30** (1993) 597; R. Narayanan  
and H. Neuberger, *Phys. Lett.* **B302** (1993) 62,  
*Phys. Rev. Lett.* **71** (1993) 3251, *Nucl. Phys.*  
**B412** (1994) 574, *Nucl. Phys.* **B443** (1995) 305;  
Y. Shamir, *Nucl. Phys.* **B406** (1993) 90.
43. H. Neuberger, *Phys. Lett.* **B417** (1998) 141,  
*Phys. Lett.* **B427** (1998) 353.
44. P. Hernandez, review talk presented at this Con-  
ference, arXiv:hep-lat/0110218.
45. V. Cirigliano and E. Golowich, *Phys. Lett.* **B475**  
(2000) 351, hep-ph/9912513.
46. V. Cirigliano, J. Donoghue and E. Golowich  
*JHEP* **0010** (2000) 048.
47. T.N. Truong, *Phys. Lett.* **B207** (1988) 495.
48. N. Isgur *et al.*, *Phys. Rev. Lett.* **64** (1990) 161  
and *Nucl. Phys.* **A508** (1990) 371.
49. E. Pallante and A. Pich, *Phys. Rev. Lett.* **84**  
(2000) 2568 and *Nucl. Phys.* **B592** (2001) 294.
50. A. J. Buras *et al.*, *Phys. Lett.* **B480** (2000)
80. M. Buchler *et al.*, hep-ph/0102287 and hep-  
ph/0102289.
51. L. Maiani and M. Testa, *Phys. Lett.* **B245** (1990)  
585.
52. D. Becirevic *et al.*, *JHEP* **0101** (2001) 012.
53. T. Blum *et al.* the RBC Collaboration,  
*Nucl. Phys. (Proc. Suppl.)* **94** (2001) 291.
54. A. Ali Khan *et al.*, the CP-PACS Collaboration,  
hep-lat/0105020.
55. M. Ciuchini *et al.*, *JHEP* **9810** (1998) 008, hep-  
ph/9808328 and refs. therein.
56. C. W. Bernard, *Nucl. Phys. (Proc. Suppl.)* **94**  
(2001) 159, hep-lat/0011064.
57. M. Lüscher, *Commun. Math. Phys.* **104** (1986)  
177; *Commun. Math. Phys.* **105** (1986) 153;  
*Nucl. Phys.* **B354** (1991) 531; *Nucl. Phys.* **B364**  
(1991) 237.
58. John F. Donoghue *et al.* *Phys. Lett.* **B119** (1982)  
412.
59. M. Ciuchini *et al.*, *JHEP* **0107** (2001) 013.
60. J. F. Donoghue and E. Golowich, *Phys. Lett.*  
**B478** (2000) 172, V. Cirigliano *et al.*, hep-  
ph/0109113.
61. M. Knecht *et al.*, S. Peris and E. de Rafael,  
*Phys. Lett.* **B457** (1999) 227 and *Phys. Lett.*  
**B508** (2001) 117.
62. T. Hambye *et al.*, *Phys. Rev.* **D58** (1998) 014017.
63. J. Bijnens *et al.*, hep-ph/0108240.
64. A. Ali Khan *et al.*, CP-PACS Collaboration, hep-  
lat/0105020.
65. D. Becirevic *et al.*, *Phys. Lett.* **B487** (2000) 74.
66. S. Peris and E. de Rafael, *Phys. Lett.* **B490**  
(2000) 213.
67. C. Bernard, *Nucl. Phys. (Proc. Suppl.)* **17** (1990)  
495.
68. G. Martinelli, *Nucl. Phys. (Proc. Suppl.)* **17**  
(1990) 523.
69. T. Bhattacharya *et al.*, arXiv:hep-lat/0111004.